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Abstract

This paper discusses some methodological approaches to model competition in a demand-based optimization framework which is particularly suitable for oligopolistic markets. Such approaches allow us to analyze oligopolies from three perspectives: at a customer level, by using discrete choice models to take into account preference heterogeneity and to model individual decisions; at an operator level, by solving a mixed integer linear program that maximizes any relevant objective function; and at a market level, by investigating the concept of Nash equilibrium in the resulting non-cooperative multi-leader-follower game. At the latter level, in order to find equilibrium solutions for this type of problem, we propose an algorithmic approach based on the fixed-point iteration method and a mixed integer linear program that enumerates the pure strategy Nash equilibria of a finite game.

Keywords

competition, equilibrium, game theory, mixed integer linear programming

1 Introduction

Oligopolistic competition occurs in various markets when a small number of operators compete for the same pool of customers. This is often the case in the transportation sector, due to reasons such as external regulations, limited capacity of the infrastructure and difficulty in entering a well-established market. Airlines are probably the most studied case of oligopoly in transportation, but intercity train operators and long-distance bus companies also operate in an oligopolistic condition in an increasing number of cases.

In transport oligopolies, operators take the supply-side decisions (e.g. price, capacity, frequency, etc.) that optimize their own objective function, which is generally related to profit maximization. Such decisions are influenced both by the preferences of the customers, who are considering to purchase one of the services available on the market, and by the decisions of the competitors.

In our work, the preferences of the customers are modelled at a disaggregate level according to the random utility theory. Using a disaggregate approach that accounts for heterogeneous needs on the demand side allows to better model supply-demand interactions. As similarly done in Pacheco Paneque *et al.* (2017), the customer utility functions are embedded in a mixed integer linear program that optimizes the operator's objective function. Competition among market players is modelled explicitly as a non-cooperative game in which all operators optimize their own decisions based on the decisions of the other operators.

The objective of our research is to analyze equilibrium solutions for such demand-based optimization models. Starting from the definition of Nash equilibrium, that is, a stationary state of the system in which no competitor has an incentive to change its decisions, we aim at examining different formulations in order to build up a general framework that is both mathematically sound and computationally tractable, and that can therefore be applied to real-life case studies.

The rest of the paper is organized as follows. Section 2 contains the literature review of game-theoretic approaches to study competition in the transport sector. Section 3 describes the two problem formulations that we have analyzed so far and presents some preliminary tests which allow us to identify the strengths and weaknesses of the current formulations. Finally, section 4 discusses the future directions of this research.

2 Literature review

The discipline that studies competition between groups of decision-makers when individual choices jointly determine the outcome is known as game theory. The reader is directed to Osborne and Rubinstein (1994) for an overview of the principal game theory concepts. For the purpose of our discussion, the key concept that needs to be introduced is Nash equilibrium (Nash, 1951). We define as Nash equilibrium of a non-cooperative game a state in which no players can improve his objective by unilaterally changing his decision. Nisan *et al.* (2007) provide an overview of several algorithmic methods used to find Nash equilibria under different circumstances.

This section presents the most relevant contributions in the field of modelling competition between operators in the transportation sector. From an historical perspective, air transportation is the area that has originated the largest amount of research. However, in recent years, thanks to regulatory changes that were passed in several countries, researchers' interest in other competitive transport markets has increased, together with the desire to study both competitive and collaborative interactions between different transport modes (see for example Adler *et al.* (2010), Behrens and Pels (2012), Hsu *et al.* (2010)).

Fisk (1984) presents various problems within transportation systems planning and operations that fit the frameworks of Nash non-cooperative games (Nash, 1951) or Stackelberg games (Von Stackelberg, 1934). The difference between these two games lies in a player's knowledge of the other players' objective function. In a two-player Stackelberg game, one player (the leader) is assumed to know the best response of the other player (the follower) to all strategies. Equivalent Stackelberg problems are frequent in transportation when a single supplier or regulator (the leader) knows the utility functions of all potential customers, who collectively play the follower role. On the other hand, in a Nash non-cooperative game there is no prior knowledge of the other player's function. For this case, the resulting game is modelled as a multi-period iterative algorithm that solves the fixed-point problem and that can converge or not to an equilibrium solution.

From an operator perspective, in the airline industry competition has been the subject of extensive studies since before the 1978 United States Airline Deregulation Act. The book by Belobaba *et al.* (2015) provides an extensive discussion on airline markets and on the effects of competition on the industry as a whole. Douglas and Miller (1974) discuss how, in a price-constrained market, equilibrium is reached through a vigorous nonprice competition focusing on the quality of the service. Frequency shares are identified as the main parameters, affecting market shares in a non-linear fashion and leading to equilibria that require the existence of only few competitors. Moving to unregulated airline markets, Panzar (1979) studies how equilibria and social welfare are related under the assumption of free entry in the market. The conclusion is that there is a divergence between welfare optimum and Nash equilibrium, since the latter results in higher prices and frequencies. Airline facility location problems are another class of problems which have been studied through a game-theoretic approach. Hansen (1990) presents a model of airline hub competition in which market shares are estimated with a logit model and are integrated with an airline cost model and an average fare model. The issue of entering a competitive airline market is analyzed by Reiss and Spiller (1989) and Berry (1992), where the relation between firms access to airports and the type of competition existing on specific origin-destination pairs is studied. Both papers highlight that, while price deregulation helped removing barriers to entry on specific routes, competition could still be hindered by route-specific or airport-specific conditions. In particular, the hub-and-spoke form of the traditional airline networks provides airlines with a competitive advantage on many routes due to the dominant position they conserve in their hub airport. In this respect, Oum *et al.* (1995) develop a framework to analyze oligopolistic competition in network-oriented markets and demostrate that incumbent airlines can use hubbing to prevent other firms from entering the market. Additionally, when applying the model to a small duopolistic case study, the authors show that hubbing is a dominant strategy, as it yields a higher profit than a point-to-point network when the rival uses a point-to-point network. However, this leads to a prisoner's dilemma equilibrium solution of the game when both players choose the hubbing strategy.

While the early research on airline competition concentrated on the supply side and only implicitly considered the demand side, in the last 15-20 years more and more attention has been dedicated to modelling passenger demand and how that affects market shares and revenues. Pels *et al.* (2000) analyze optimal airfares, frequencies and passenger charges in a large metropolitan area that hosts more than one airport. To model airport and airline competition, the authors model passenger choice with a nested multinomial logit model (NMNL) and include the derived choice probabilities in the airlines' maximization function. In Adler (2001), competition in a deregulated air transport market is modelled by means of a two-stage Nash best response game. In the first stage, an integer linear program is used to model the simultaneous choice airlines make to choose their hub-and-spoke network. In the second stage, airlines aim at maximizing profits by choosing fares, frequencies and aircraft sizes for their network of routes. Passenger choices are affected by airline decisions and are estimated by using a multinomial logit model (MNL), from which market shares are derived. This results in a nonlinear mathematical program to be solved for all competitors iteratively until either a sub-game Nash equilibrium or a quasi-equilibrium solution (i.e. two or more possible solutions around which the program cycles) are found. The open questions are related to the uniqueness of the Nash equilibrium solution found by the iterative algorithm, to the relation between the concepts of Nash equilibrium and quasi-equilibrium, and to the possibility of not reaching equilibrium or convergence through the iterative process. In Adler *et al.* (2010), a competitive game that includes high-speed rail and air transport options is studied to evaluate the potential economic impact of the Trans-European high-speed rail network. As in Adler (2001), schedules and prices are endogenous variables in the model, which has a non-linear objective function. Finally, Aguirregabiria and Ho (2012) study a dynamic version of the airline network design game, in which airlines decide for every period which non-stop flights to operate and the proposed fares. A nested logit model is used to predict the market shares that are used to estimate airline profits with the proposed prices.

In all the above-mentioned papers, modelling demand by using discrete choice models results in a non-linear optimization model. A new framework that can integrate any type of discrete choice model in a mixed integer linear program is introduced in Pacheco Paneque *et al.* (2017), where the issues of non-linearity and non-convexity caused by probabilistic choice models are tackled by relying on simulation. Such formulation constitutes the starting point of our research and is described in more details in section 3.1. Our objective is to extend this framework to be able to model not only the Stackelberg game played between an operator and its potential customers, but also the competition between operators in an oligopolistic market.

3 Problem formulations

In this section, we first introduce the demand-based optimization model for the single operator case, which is largely inspired by Pacheco Paneque *et al.* (2017). Then, we propose two solution approaches that extend this model to take into account the competitive case involving more than one operator.

3.1 The optimization model for the single operator

The problem we consider belongs to the class of problems known as bilevel optimization problems. In such problems there is a hierarchical relationship between two autonomous groups of decision-makers, each trying to optimize its own objective function (Colson *et al.,* 2007). In particular, the decisions taken in the upper-level optimization problem affect the decisions taken in the lower-level optimization problem. In the formulation that follows, the supplier's pricing and availability strategy for the different alternatives affects the customers' decision. Under the assumption that the supplier knows the utility functions of all customers, this is equivalent to defining a Stackelberg game with the supplier playing as leader and the customers playing as followers.

Customers are modelled as utility maximizers and always select the alternative that gives them the highest utility. According to the random utility theory, the customer utility function is composed of a deterministic part, which captures all the observable variables, and of a random component, which takes into account unobserved attributes and taste variations. Due to such random component, customer choice is probabilistic and depends on the assumed distribution of the error term of the utility function. As a consequence, choice model are generally non-linear. In order to be able to integrate them in a mixed integer linear program, Pacheco Paneque *et al.* (2017) propose a framework in which simulation is used to generate random draws from the error term distribution. For each customer and alternative, each draw corresponds to a behavioral scenario. In each scenario, customers choose deterministically the available alternative with the highest utility. However, over a sufficiently high number of behavioral scenarios, the choice distribution across the alternatives can approximate the probabilities derived from the estimation of the discrete choice model.

The following paragraphs explain the notation for the sets, parameters and decision variables used in the model.

Let I be the set of available alternatives in the considered market. The set $I = \{0, 1, .., c\}$ includes a number c of alternatives offered by an operator and an opt-out alternative 0. Each alternative $i \in I$ has a capacity C_i. Additionally, let N be a population of customers. Each customer $n \in N$ considers a choice set $I_n \subseteq C$ and associates a utility to each alternative $i \in I_n$. The deterministic part of the utility depends on the endogenous variables, which characterize the interactions between demand and supply, and on the exogenous demand variables, which are given and are therefore treated as parameters in the optimization model. In the rest of the paper we assume that price is the only endogenous variable in the model. If we define R as the set of behavioral scenarios, then for each scenario $r \in R$, customer $n \in N$ and alternative $i \in I$ we have that ξ_{inr} is the random error term drawn from the distribution. Additionally, β_{in} is the parameter associated with the endogenous price variable and q_{in} is the exogenous term. Also, the order in which customers have access to the alternatives is defined exogenously with a priority list, as in Binder *et al.* (2017), and L_{in} is a value that determines the position of customer $n \in N$ for alternative $i \in I$.

There are two types of upper-level decision variables: (1) the operator availability variables y_{in}, binary variables that are equal to 1 if alternative $i \in I$ is offered to customer $n \in N$ and are equal to 0 otherwise; (2) the continuous price variables p_{in}, which indicate the price at which alternative $i \in I$ is offered to customer $n \in N$. For modelling purposes, lower bounds lbP_{in} and upper bounds ubP_{in} are defined for the price variables, which in turn define lower bounds lbU_{nr} and upper bounds ubU_{nr} for the utilities for each customer $n \in N$ and scenario $r \in R$.

In the model there are also the following lower-level decision variables: (3) the customer availability variables y_{inr}, binary variables that are equal to 1 if alternative $i \in I$ is available to customer $n \in N$ in scenario $r \in R$ and are equal to 0 otherwise; (4) the customer choice variables w_{inr}, binary variables that are equal to 1 if in scenario $r \in R$ customer $n \in N$ selects alternative $i \in I$ and are equal to 0 otherwise; (5) the linearized price-choice variables α_{inr}, which are equal to p_{in} if $w_{inr} = 1$ and are equal to 0 otherwise; (6) the utility variables U_{inr}; (7) the discounted utility variables z_{inr}, which are equal to U_{inr} if $i \in I$ is available to $n \in N$ in scenario $r \in R$ and are equal to lbU_{nr} otherwise; (8) the variables U_{nr} that define the maximum utility for customer $n \in N$ in scenario $r \in R$, i.e. $U_{nr} = \max_{i \in I} z_{inr}$

With such definitions, the mathematical model is then formulated as follows:

$$\max \quad \frac{1}{R} \sum_{i \in I \setminus \{0\}} \sum_{n \in N} \sum_{r \in R} \alpha_{inr} \tag{1}$$

s.t.

$$\sum_{i \in I} w_{inr} = 1 \qquad \forall n \in N, \forall r \in R \tag{2}$$

$$w_{inr} \leq y_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{3}$$

$$y_{inr} \leq y_{in} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{4}$$

$$y_{in} = 0 \qquad \forall i \in I, \forall n \in N : i \notin C_n \tag{5}$$

$$\sum_{n \in N} w_{inr} \leq C_i \qquad \forall i \in I \setminus \{0\}, \forall r \in R \tag{6}$$

$$C_i(y_{in} - y_{inr}) \leq \sum_{m \in N : L_{im} < L_{in}} w_{imr} \qquad \forall i \in I \setminus \{0\}, \forall n \in N, \forall r \in R \tag{7}$$

$$\sum_{m \in N : L_{im} < L_{in}} w_{imr} \leq (C_i - 1) y_{inr} + (n - 1)(1 - y_{inr}) \qquad \forall i \in I \setminus \{0\}, \forall n \in N, \forall r \in R \tag{8}$$

$$U_{inr} = \beta_{in} p_{in} + q_{in}^d + \xi_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{9}$$

$$lbU_{nr} \leq z_{inr} \leq lbU_{nr} + M_{U_{nr}} y_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{10}$$

$$U_{inr} - M_{U_{nr}}(1 - y_{inr}) \leq z_{inr} \leq U_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{11}$$

$$z_{inr} \leq U_{nr} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{12}$$

$$U_{nr} \leq z_{inr} + M_{U_{nr}}(1 - w_{inr}) \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{13}$$

$$lbP_{in} \leq p_{in} \leq ubP_{in} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{14}$$

$$lbP_{in} w_{inr} \leq \alpha_{inr} \leq ubP_{in} w_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{15}$$

$$p_{in} - (1 - w_{inr}) ubP_{in} \leq \alpha_{inr} \leq p_{in} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{16}$$

The objective function (1) maximizes the total revenue of the operator, obtained by summing up all the prices paid by the customers to access the chosen service. Constraints (2) ensure that a customer chooses only one alternative in each scenario. Constraints (3) state that an alternative can be chosen only if it is available to the customer in a specific scenario, while constraints (4-5) guarantee that, when an alternative is either (a) not offered to the customer by the operator or (b) is not included in the customer's choice set, then it is unavailable in all scenarios. Constraints (6-8) are the capacity constraints, which take into account the exogenous priority lists, i.e. the order in which customers are assumed to access to the market. Constraints (9) define the values of the utility function for each alternative-customer-scenario triplet. Constraints (10-13) make sure that, for each scenario realization, each customer always chooses the available alternative associated to the highest utility. Constraints (14) define the bounds within which the prices can be set by the operator. Finally, constraints (15-16) are used to derive the value of the linearized price-choice variables from the price and the choice variables.

The validity of this model for the purpose of our analysis is twofold. In fact, it can be used to analyze the decisions of a single operator while holding the other operators' decisions fixed, but it can also model a *collaborative* setting in which operators act with a cartel-like behavior aiming at maximizing their collective profit. In the latter case, the solution value provides an upper bound for the sum of the operator profits in the *competitive* setting, which could be useful both to improve computational times and to study the effects of competition on producer and consumer surplus.

3.2 Solution approaches for the multiple operators game

In oligopolistic markets there are multiple operators that simultaneously solve an optimization problem like the one introduced in the previous paragraph. The result is a non-cooperative multi-leader-follower game in which each leader solves a mathematical program with equilibrium constraints (MPEC). This category of problems has been the subject of extensive research in recent years. Relevant contributions include but are not limited to Luo *et al.* (1996), Ferris and Pang (1997), Pang and Fukushima (2005) and Leyffer and Munson (2010). If compared to the topics covered in these works, the peculiarity of our problem lies in the integrality constraints enforced on the lower-level variables that model customer choice according to the utility maximization framework. The goal of our research is to investigate the concept of Nash equilibrium for this type of games.

In the work done so far, we have followed two lines of research. In the first one, we examine an algorithmic approach based on the fixed-point iteration method, which sequentially solves the single operator optimization model presented in Section 3.1. In the second one, we assume that all operators have a finite set of pure strategies from which they simultaneously select a pure or mixed strategy.

3.2.1 A fixed-point iteration method

The fixed-point iteration method has been used in several occasions to search for Nash equilibrium solutions in competitive transport markets. Examples mentioned in section 2 include Fisk (1984) and Adler (2001). Algorithm 1 describes our implementation of the fixed-point iteration method. The notation applies to a case of duopolistic competition, but it can be easily modified to take into account problems involving more than two competitors. In the initialization phase (lines 2-6), two heuristic decisions must be taken to define the first optimizing operator and an initial feasible strategy for the competitor. In the iterative phase (lines 7-16), operators take turns and each of them plays its best response pure strategy to the last strategy played by the competitor. Such sequential game terminates when one of the operators selects a strategy that it already played in one of the previous iterations (line 10), as it would induce the same sequence of best responses as before. The solution of this game can be either a Nash equilibrium for the game or a set of n strategies for each player, with $n > 1$, which would continue to be played cyclically (lines 17-22).

A numerical example

This section presents some preliminary numerical experiments aimed at understanding the properties of the fixed-point iteration method. The case study used for the tests is derived from Ibeas *et al.* (2014), where the choice of customers among three different parking alternatives is modelled by means of a disaggregate choice model. In our work we assume that two of these alternatives (paid underground parking and paid on-street parking) are managed by two competing operators, while the third alternative (free on-street parking) is considered as the opt-out option, since it does not provide any revenue to the operators. We further assume that each operator decides on a unique price to be proposed to all customers. The test instance has the following size: 2 competitors, 3 alternatives (1 alternative per operator and 1 opt-out alternative), 50 customers, 5 behavioral scenarios.

The main comments on the performed tests are the following: (i) the fixed-point iteration method is effective in either finding one Nash equilibrium solution of the game or reaching a quasi-equilibrium where the program cycles around a number of possible solutions; (ii) however, since the strategy set of each player is not finite due to the continuous price variables, we

Algorithm 1 Fixed-point algorithm

```
procedure Fixed-point Algorithm
    iter ← 1
    cycle ← 0
    opt_iter ← 1
    nonOpt_iter ← 2
    y_{iter,nonOpt_iter} ← a feasibile strategy for player 2
    while cycle = 0 do
        y_{iter,opt} ← SingleOperatorMILP(y_{iter,nonOpt})
        for i from 1 to iter do
            if y_{iter,opt} = y_{i,opt} & opt_iter = opt_i then
                cycle ← 1
                startCycle ← i
        if cycle = 0 then
            iter ← iter + 1
            opt_iter ← nonOpt_{iter−1}
            nonOpt_iter ← opt_{iter−1}
    if iter − i = 2 then
        equilibriumSolution ← (y_{iter,opt}, y_{iter−1,nonOpt})
    else
        quasiEquilibriumSolution ← ∅
        for i from startCycle to iter − 1 do
            quasiEquilibriumSolution ← quasiEquilibriumSolution ∪ y_{i,opt_i}
```

cannot guarantee that a Nash equilibrium solution exists for any given problem and, if it exists, that it is unique; (iii) different initial strategies can lead to different equilibria of the game, and no conclusion can be drawn on the relationship between such initial strategies and their corresponding equilibria.

Figure 1 shows how the decisions of the operators change throughout the sequential process and how they affect the operators' profits. In the specific case considered, a cycle which includes 5 support strategies for each player is reached, which means that the values found between iteration 4 and 14 would be repeated again and again if the game was to continue. On the left graph, we can notice how for both operators the price decisions quickly converge towards a "desirable" region, which in this case is close to the optimal values of the collaborative solution. On the right graph, we can see the corresponding payoffs and we can graphically quantify the effect of competition on profits as the difference between the sum of the firms' profits when behaving as a cartel and the sum of the their profits in the competitive framework.

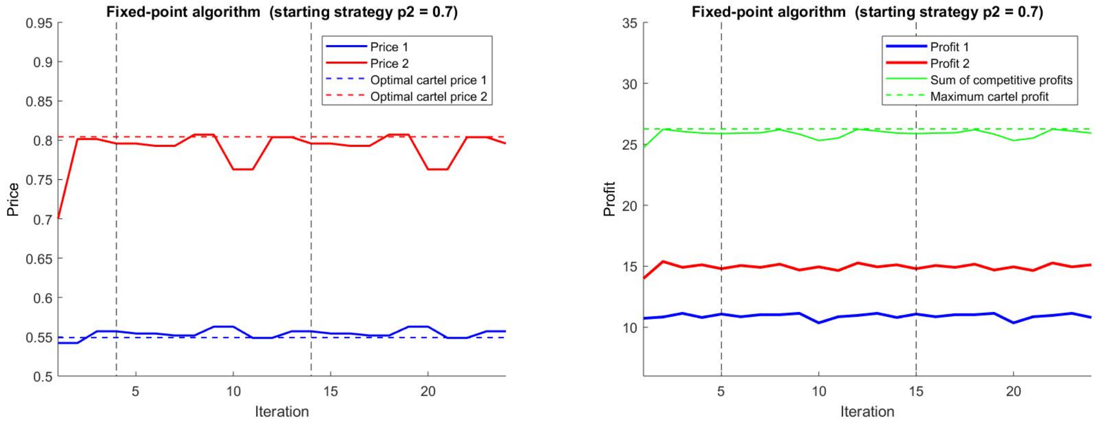

Figure 1: Prices and revenues as a function of the iterations.

While the fixed-point iteration method is an efficient method to find a Nash equilibrium solution of the game, it still leaves some questions unanswered (see section 4). In particular, the uncertainty about the existence of a Nash equilibrium and the fact that different initial solutions can lead to different equilibria constitute notable limitations of this approach.

3.2.2 Towards a simultaneous optimization model

Another way to search for Nash equilibria in the solution space is to consider a finite set of strategies for each operator. To do so, we define discrete price levels instead of the continuous price variables used in section 3.1. Since now both upper-level decision variables are discrete, we have a finite game. Nash (1951) proves that every finite game has at least one mixed strategy equilibrium point, but no equivalent result is available when considering pure strategies. Well-known algorithms to find Nash equilibria include the Lemke-Howson algorithm (Lemke and Howson, 1964) and the Porter-Nudelman-Shoham algorithm (Porter *et al.,* 2008), while a mixed integer program formulation to find Nash equilibria is introduced in Sandholm *et al.* (2005). The latter paper proposes a model whose feasible solutions coincide with the equilibria of the game and which allows to specify an objective, such as social welfare maximization, to be optimized over the solution space. It has been demonstrated that the problem of finding Nash equilibria belongs to the complexity class PPAD (Daskalakis *et al.,* 2009).

Let us consider a two-player game, in which $K = 1, 2$ is the set of players. Each player has a finite set of pure strategies S_k. Then we can define the set of pure strategy solutions of the game as the Cartesian product $S = \prod_{k \in K} S_k$. For each solution $s \in S$, we can derive a payoff function U_{ks} for each operator $k \in K$.

A pure strategy Nash equilibrium of a two-player game is a solution such that both players play their best response pure strategy to the given strategy of the competitor. A trivial but computationally expensive method to find pure strategy Nash equilibrium solutions of a finite game is enumerating all the solutions of the game and then checking whether the best response condition is verified for all players. Our ultimate goal is to develop an efficient method to explore the solution space, possibly by extending the analysis to take into account mixed strategies. As of now, however, we have only formulated an enumeration model which does not perform any actual optimization. Nevertheless, we present the current version of the model, which shows how to translate an enumeration problem in a mixed integer program and how it is possible to integrate this in the existing demand-based optimization framework.

Here we first introduce the notation and then we present and explain the mathematical model.

Sets

N	Population of customers.
K	Set of operators.
I	Set of all available choices.
I_n	Set of choices considered by customer $n \in N$. $I_n \subseteq I$
I_k	Set of alternatives operated by operator $k \in K$. $I_k \subseteq I$
R	Set of behavioral scenarios (draws)
S_k	Set of possible strategies for operator $k \in K$
S	Set of possible solutions of the game. $\lvert S \rvert = \prod_{k \in K} \lvert S_k \rvert$
S_k^C	Set of strategies that can be played against $k \in K$. $\lvert S_k^C \rvert = \prod_{l \in K \setminus \{k\}} \lvert S_l \rvert$

Parameters

C_i	Capacity of alternative $i \in I \setminus \{0\}$
L_{in}	Integer value indicating the position in the priority list of customer $n \in N$ for alternative $i \in I$
lbU_{nr}	Lower bound for the utility for customer $n \in N$ in scenario $r \in R$
ubU_{nr}	Upper bound for the utility for customer $n \in N$ in scenario $r \in R$
$M_{U_{nr}}$	Value used in the big-M constraints
β_{in}	Parameter associated with the price component of the utility function
q_{in}	Exogenous term of the choice model for service i and individual n
ξ_{inr}	r-th draw from the error term of the choice model for service i and individual n
p_{ins}	Price of $i \in I \setminus \{0\}$ for costumer $n \in N$ in solution $s \in S$
y_{ins}	Availability of alternative $i \in I$ for costumer $n \in N$ in solution $s \in S$

Variables

y_{inrs}	Binary variable equal to 1 if $i \in I$ is available to $n \in N$ in scenario $r \in R$ and solution $s \in S$ and 0 otherwise
w_{inrs}	Binary variable equal to 1 if $i \in I$ is chosen by $n \in N$ in scenario $r \in R$ and solution $s \in S$ and 0 otherwise
U_{inrs}	Utility associated to alternative $i \in I$ by customer $n \in N$ in scenario $r \in R$ and solution $s \in S$.
z_{inrs}	Discounted utility, is U_{inr} if $i \in I$ is available to $n \in N$ in scenario $r \in R$ and solution $s \in S$ and lbU_{nr} otherwise
U_{nrs}	Maximum utility for customer $n \in N$ in scenario $r \in R$ and solution $s \in S$. $U_{nrs} = \max_i z_{inrs}$
V_{ks}	Utility for operator $k \in K$ in solution $s \in S$
V^{max}_{kt}	Utility for the best response of operator $k \in K$ to the competitor's strategy $t \in S^C_k$
x_{ks}	Binary variable equal to 1 in solution $s \in S$ if operator $k \in K$ is plays its best response strategy and 0 otherwise
e_s	Binary variable equal to 1 if the solution $s \in S$ is a Nash equilibrium for the game and 0 otherwise

With such definitions, the mathematical model can then be formulated as follows:

Find $s \in S$ such that $e_s = 1$

$s.t.$

Equilibrium constraints:

$$e_s \geq \sum_{k \in K} x_{ks} - (|K| - 1) \qquad \forall s \in S \qquad (17)$$

$$e_s \leq x_{ks} \qquad \forall k \in K, \forall s \in S \qquad (18)$$

Operator constraints:

$$V_{ks} = \frac{1}{R} \sum_{i \in C_k} \sum_{n \in N} \sum_{r \in R} p_{ins} w_{inrs} \qquad \forall k \in K, \forall s \in S \qquad (19)$$

$$V_{ks} \leq V^{max}_{kt} \qquad \forall k \in K, \forall s \in S_k, \forall t \in S^C_k \qquad (20)$$

$$V^{max}_{kt} \leq V_{ks} + M_r(1 - x_{ks}) \qquad \forall k \in K, \forall s \in S_k, \forall t \in S^C_k \qquad (21)$$

$$\sum_{s \in S} x_{ks} = \left|S^C_k\right| \qquad \forall k \in K \qquad (22)$$

Customer constraints:

$$\sum_{i \in I} w_{inrs} = 1 \qquad \forall n \in N, \forall r \in R, \forall s \in S \qquad (23)$$

$$w_{inrs} \leq y_{inrs} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in S \qquad (24)$$

$$y_{inrs} \leq y_{ins} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in S \qquad (25)$$

$$y_{ins} = 0 \qquad \forall i \in I, \forall n \in N : i \notin C_n, \forall s \in S \qquad (26)$$

$$\sum_{n \in N} w_{inrs} \leq C_i \qquad \forall i \in I \setminus \{0\}, \forall r \in R, \forall s \in S \qquad (27)$$

$$C_i(y_{ins} - y_{inrs}) \leq \sum_{m \in N : L_{im} < L_{in}} w_{imrs} \qquad \forall i \in I \setminus \{0\}, \forall n \in N, \forall r \in R, \forall s \in S \qquad (28)$$

$$\sum_{m \in N: L_{im} < L_{in}} w_{imrs} \leq (C_i - 1) y_{inrs} + (n-1)(1 - y_{inrs}) \qquad \forall i \in I \setminus \{0\}, \forall n \in N, \forall r \in R, \forall s \in S \quad (29)$$

$$U_{inrs} = \beta_{in} p_{ins} + q^d_{in} + \xi_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in S \quad (30)$$

$$lbU_{nr} \leq z_{inrs} \leq lbU_{nr} + M_{U_{nr}} y_{inrs} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in S \quad (31)$$

$$U_{inrs} - M_{U_{nr}}(1 - y_{inrs}) \leq z_{inrs} \leq U_{inrs} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in S \quad (32)$$

$$z_{inrs} \leq U_{nr} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in S \quad (33)$$

$$U_{nr} \leq z_{inrs} + M_{U_{nr}}(1 - w_{inrs}) \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in S \quad (34)$$

Constraints (17-18) ensure that a strategic scenario is a Nash equilibrium for the game if and only if each operator is playing the best response to the opponent's strategy. Constraints (19) define the utility for each operator in each strategic scenario. Constraints (20-22) make sure that, for each opponent strategy, an operator always chooses one best response strategy. Constraints (23-34) maintain the original meaning of constraints (3-13) and are applied to all strategic scenarios: constraints (23) ensure that a customer chooses only one alternative in each behavioral scenario; constraints (24-26) relate the choice variables with the availability variables; constraints (27-29) are the capacity constraints; constraints (30) define the values of the utility function; constraints (31-34) ensure that customers always chooses the available alternative associated to the highest utility.

A numerical example

This section presents an example of the results that can be obtained by enumerating the solutions of a pure strategy. The case study introduced in section 3.2.1 is also used here. The instance was modified to take into account discrete pricing strategies. The range of the prices was selected in accordance with the results of the fixed-point iteration method, which identifies a region of the solution space around which equilibrium solutions should be located.

The main comments on the performed tests are the following: (i) when considering a finite game, it is possible to find all pure strategy Nash equilibria by enumerating all possible solutions of the game and then deriving the payoff matrices; (ii) different support strategies can lead to different pure strategy Nash equilibria of the resulting game; (iii) to decide whether the assumption of a finite game is realistic or not, it is necessary to take into account the features of the studied problem. Using transportation examples, it is clear that the assumption of price discretization is easier to justify in low-price markets like parking and local public transport fares than in the airline industry.

Figure 2 shows the payoff matrices for two two-player games built from the parking case study, Payoff matrix of player 1

S1 \ S2	*0,70*	*0,73*	*0,76*	*0,79*	*0,82*	*0,85*
0,50	10,00	10,00	10,00	10,00	10,00	10,00
0,53	10,49	**10,60**	**10,60**	10,60	10,60	10,60
0,56	**10,53**	10,42	10,53	**10,86**	**11,20**	11,20
0,59	10,27	10,03	9,80	9,91	10,62	**11,45**
0,62	10,04	9,80	9,42	9,42	9,42	9,92
0,65	9,62	9,36	8,84	8,45	8,71	8,58

Payoff matrix of player 2

S1 \ S2	*0,70*	*0,73*	*0,76*	*0,79*	*0,82*	*0,85*
0,50	14,00	14,45	14,74	14,69	**14,76**	14,62
0,53	14,00	14,45	14,74	**15,01**	14,60	14,45
0,56	14,00	14,60	14,74	**14,85**	14,76	14,28
0,59	14,00	14,60	15,05	**15,48**	15,09	14,45
0,62	14,00	14,60	15,20	15,48	**15,91**	15,81
0,65	14,00	14,60	15,20	15,80	15,91	**16,32**

(a) Game with 1 pure strategy Nash equilibrium

Payoff matrix of player 1

S1 \ S2	*0,75*	*0,77*	*0,79*	*0,81*	*0,83*	*0,85*
0,50	10,00	10,00	10,00	10,00	10,00	10,00
0,52	10,40	10,40	10,40	10,40	10,40	10,40
0,54	**10,80**	**10,80**	10,80	10,80	10,80	10,80
0,56	10,42	10,53	**10,86**	**11,09**	**11,20**	11,20
0,58	9,74	9,86	10,09	10,44	10,67	**11,37**
0,60	9,60	9,60	9,72	10,08	10,44	10,68

Payoff matrix of player 2

S1 \ S2	*0,75*	*0,77*	*0,79*	*0,81*	*0,83*	*0,85*
0,50	14,70	**14,78**	14,69	14,74	14,28	14,62
0,52	14,70	**15,09**	14,85	14,58	14,61	14,45
0,54	14,85	14,94	**15,17**	14,74	14,44	14,45
0,56	14,85	**14,94**	14,85	14,90	14,61	14,28
0,58	15,00	15,09	**15,17**	15,07	15,11	14,45
0,60	15,00	15,25	**15,48**	15,39	15,27	14,30

(b) Game with no pure strategy Nash equilibrium

Figure 2: Payoff matrices for two games with different support strategies. Best response payoffs are in bold. Equilibrium payoffs are in blue.

which differ only in the set of support strategies of the operators. We can notice that the same solution ($S = (0.56, 0.79)$) is a Nash equilibrium in game (a), but is not a Nash equilibrium in game (b), since in the second case player 2's best response to player 1's strategy $p_1 = 0.56$ is to play $p_2 = 0.77$.

4 Conclusions and future research directions

In this paper, we presented different methodological approaches to model competition in a demand-based optimization framework. Our research aims at analyzing oligopolistic markets from three integrated perspectives: (i) at a customer level, using discrete choice models allows to take into account preference heterogeneity and to model individual decisions according to the utility maximization principle; (ii) at an operator level, solving the corresponding optimization problem allows to understand how the strategic decisions are affected both by the knowledge of the customers' utility functions and by the existence of competitors with conflicting interests; (iii) at a market level, formulating the problem as a non-cooperative multi-leader-follower game played by operators and customers allows to investigate the concept of Nash equilibrium for the given market.

Starting from a recent mixed integer linear program that models the Stackelberg game played between a single operator and the customers, we propose two formulations that incorporate competition in the existing framework in order to study the equilibria of the resulting game. The first method is based on the fixed-point iteration algorithm and can be applied to both continuous and discrete games. The algorithm succeeds in finding pure strategy Nash equilibrium or quasi-equilibrium solutions in our test instances. However, it is not possible to prove that a Nash equilibrium solution exists and, if it exists, that it can be found or that it is unique. The second method relies on solution enumeration and can therefore be applied only to small finite games. Pure strategy Nash equilibrium solutions can be found when they exist, however Nash's existence theorem only guarantees the existence of at least one mixed strategy equilibrium for such finite games.

Future directions of our research will address the following open questions. We will initially consider the extension of the current formulations to include mixed strategy games. Furthermore, given the non-linearity and non-convexity of the payoff functions caused by the customer choice variables, we would like to determine whether an efficient strategy to search for equilibria in the solution space of the enumeration model can be found. Finally, we aim at better understanding how mathematical programs with equilibrium constraints can be helpful in addressing our problem.

To conclude, it is worth noticing that the application of game theory to modelling competition within the transportation sector cannot prescind from a problem-specific analysis of the considered market. Preliminary numerical experiments show that modelling customer choice at a disaggregate level can lead to multiple equilibrium solutions which are all located in a confined region of the solution space. Nonetheless, it is reasonable to assume that these different solutions (or a subset of them) might be perceived as a unique solution by the operator in its decision-making process, since in many real-life cases a marginal increase in profits would not justify a change in the operator's pricing strategy. If this holds true, it becomes relevant to understand whether the concept of Nash equilibrium region can also be useful to study competition in transportation.

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